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UNCONSTRAINED VARIATIONAL STATEMENTS FOR INITIAL AND BOUNDARY-VALUE PROBLEMS

T. E. Simkins

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LARGE CALIBER WEAPON SYSTEMS LABORATORY
BENÉT WEAPONS LABORATORY
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A procedure is developed for generating variational statements suitable for obtaining approximate solutions to boundary-initial value problems. The essence of the procedure is to introduce all boundary and initial conditions into the variational statement as natural boundary conditions. This is accomplished through the use of Lagrange multipliers in which all initial condition terms as well as boundary terms are determined analytically. The result is a variational statement in which completely unconstrained trial functions may be assumed as a basis for an approximate solution. Several applications are

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given, including the response of a beam subject to a moving concentrated mass loading.

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SUMMARY. A procedure is developed for generating variational statements suitable for obtaining approximate solutions to boundary-initial value problems. The essence of the procedure is to introduce all boundary and initial conditions into the variational statement as natural boundary conditions. This is accomplished through the use of Lagrange multipliers, in which all initial condition terms as well as boundary terms are determined analytically. The result is a variational statement in which completely unconstrained trial functions may be assumed as a basis for an approximate solution. Several applications are given, including the response of a beam subject to a moving concentrated mass loading.

I. INTRODUCTION. Theorems establishing the correspondence between certain boundary value problems and the calculus of variations appear in detail in the treatise of Collatz [1]. More recently, Rund [2] has produced more general theorems through the use of the transversality conditions from which the theorems of Collatz emerge as special cases. Neither of these works, however, attempts to establish variational statements for the solution of initial value problems, which is the subject of the work herein.

Recent work by Bailey [3,4] has shown that Hamilton's law of varying action is capable of yielding approximate solutions to initial and boundary-initial value problems. The variational form used by Bailey allows the function and its derivative to vary at the upper limit of integration of the time interval. At the lower limit, these quantities are constrained to satisfy specified initial conditions. It appears that a more general method would free the variations at the lower limit as well, thus broadening the class of admissible trial functions. Although much has been written [5] on the subject of removal of constraints on the boundary variations, the applications usually deal with elliptic rather than hyperbolic systems, where it is customary to introduce the constraints of the problem as natural boundary conditions. The constraints themselves thus become subject to approximation through the

variational process. Convergence, when achieved, tends toward a solution satisfying these constraint conditions. When all of the constraints of the problem are introduced in this manner, the result is a completely unconstrained variational statement, whereby the trial functions need not identically satisfy any boundary conditions. As the use of Lagrange multipliers in freeing boundary variations of constraints is by now classical, this work is fundamentally concerned with extending the method to remove constraining time conditions.

The Lagrange multiplier procedure adds each constraint as a zero times a Lagrange multiplier to the previously unconstrained variational statement. In this way, each constraint is made to appear as a natural boundary condition and, in some cases where a functional exists (i.e., variational "principles"), it may be modified to include these terms. The multipliers can usually be identified in terms of values of the function and its derivatives on the bounding surface of the domain of integration. The act of freeing the boundary variations will not result in the loss of a variational "principle" provided the constraint is holonomic, i.e., a functional will still exist though modified by additive products of the Lagrange multipliers times the individual constraint relations. On the other hand, should any of the constraints be nonholonomic, the existence of a functional is denied and one has in its place a less elegant variational "statement" which may nevertheless provide a basis for an approximate solution to the problem at hand.

In spite of the apparent generality of the Lagrange multiplier method, its application to Hamilton's principle (a constrained variational principle) for the solution of initial value problems is not obvious. Indeed, Bailey found it fruitful to employ Hamilton's law of varying action in which no functional exists. Once the quest of a functional is abandoned, however, unconstrained variational formulations for initial value problems are immediately possible, as was first shown by Tiersten [6]. Since the purpose of Tiersten's work at the time did not involve explicit solutions in the time domain, the success of his method for achieving solutions to initial value problems was never tested. Further, Tiersten's procedure requires a special introduction of one of the initial conditions into the variational statement, making incomplete use of the Lagrange multiplier method in the time domain. Solutions to the free oscillator, the two-dimensional wave equation, and the motion of a beam to a moving concentrated mass are offered as evidence of success of the method.

The variational form presented herein does not produce a functional. Gurtin [7] and later Wu [8], however, have successfully formulated initial and boundary-initial value problems using unconstrained variational principles in which a functional is indeed produced. Wu's treatment requires introducing the adjoint variable and replacing the given

boundary and initial conditions with a set of artificial conditions containing a parameter that eventually is allowed to become very large numerically. Using finite element approximations, Wu was able to achieve excellent agreement with the exact solutions of several partial differential equations in one and two dimensions. Gurtin's procedure, on the other hand, combines the initial conditions and field equations into a single integro-differential equation which is then regurgitated as the Euler-Lagrange equation when the variation of a constructed functional is made to vanish.

II. VARIATIONAL STATEMENTS AND LAGRANGE MULTIPLIERS. Unless the variational quantities appearing in a variational statement or principle are completely arbitrary, the formulation is said to be 'constrained'. The principle of virtual work in its conventional form is one example of a constrained variational principle; i.e.,

$$0 = \delta\int_V U(\varepsilon)dV - \int_V K_i\delta u_i dV - \int_{S_f} \bar{F}_i\delta u_i dS \tag{1}$$

where $U(\varepsilon)$ is the potential energy density of an elastic volume, V. The K_i are body forces per unit volume and the u_i are the unknown displacement functions. A bar denotes prescribed surface quantities, and

$$F_k = n_\ell \frac{\partial U}{\partial u_{k,\ell}} \equiv n_\ell\sigma_{\ell k} \tag{2}$$

where n_ℓ is the outward directed normal to any surface. $\sigma_{\ell k}$ represents the stress tensor.

Implicit in Eq. (1) is the constraint

$$u_i = \bar{u}_i \text{ on } Su, \tag{3}$$

i.e., the displacement functions must be those prescribed on the boundary surface Su. Further, Su and S_f do not overlap and together comprise the complete boundary of the volume, V.

If Eq. (1) is used as a basis for approximating a solution to a problem in elastostatics - e.g., via the Rayleigh-Ritz method - the shape functions employed in the approximation must each identically satisfy the constraint equation (3) a priori. This requirement may be removed by using Lagrange multipliers to introduce the constraint explicitly in Eq. (1) which then becomes

$$0 = \delta\{\int_V U(\varepsilon)dV + \int_{S_u} \lambda_i(u_i-\bar{u}_i)dS\} - \int_V K_i\delta u_i dV - \int_{S_f} \bar{F}_i\delta u_i dS \qquad (4a)$$

$$= \delta \int_V U(\varepsilon)dV + \int_{S_u} \{\delta\lambda_i(u_i-\bar{u}_i) + \lambda_i\delta u_i\}dS - \int_V K_i\delta u_i dV - \int_{S_F} \bar{F}_i\delta u_i dS \ .$$

For a Hookean material

$$\delta\int_V U(\varepsilon)dV = \int_V \sigma_{ij}\delta\varepsilon_{ij}dV$$

$$= \int_V (\sigma_{ij}\delta u_i)_{,j}dV - \int_V \sigma_{ij,j}\delta u_i dV$$

$$= \int_{S_f} F_i\delta u_i dS + \int_{S_u} F_i\delta u_i dS - \int_V \sigma_{ij,j}\delta u_i dV \ .$$

Thus,

$$\int_{S_u} F_i\delta u_i dS - \int_V (\sigma_{ij,j}+K_i)\delta u_i dV + \int_{S_u} [\lambda_i\delta u_i+\delta\lambda_i(u_i-\bar{u}_i)]dS$$

$$+ \int_{S_f} (F_i-\bar{F}_i)\delta u_i dS = 0 \ . \qquad (4b)$$

Now the δu_i in Eq. (1) or Eq. (4b) are not all arbitrary because of the constraints in Eq. (3). But if the Lagrange multipliers, λ_i, are defined as $-F_i$ on S_u, the coefficients of all δu_i quantities on S_u vanish, and hence the δu_i may be viewed as arbitrary. This is the essence of the Lagrange multiplier method. It is important to note that the constraints involved in this exercise are holonomic and the surfaces S_u and S_f do not overlap.

Thus, substituting $\lambda_i \equiv -F_i$ on S_u, Eq. (4a) becomes an unconstrained principle of virtual work:

$$\delta\{\int_V U(\varepsilon)dV - \int_{S_u} F_i(u_i-\bar{u}_i)dS - \int_V K_i u_i dV - \int_{S_f} \bar{F}_i u_i dS\} = 0 \ . \qquad (4c)$$

If the Rayleigh-Ritz method is used with Eq. (4c), the shape functions assumed no longer need satisfy identically the constraint relations in Eq. (3).

Another example of a constrained variational principle is Hamilton's Principle [6]

$$\delta\int_{t_0}^{t_1} dt\{\int_V (T-U)dV + \int_{S_f} \bar{F}_k u_k dS\} = 0 \ . \tag{5}$$

In analogy with the previous treatment of the virtual work principle one notes that instead of simply

$$\int_V U(\varepsilon)dV \ ,$$

we have

$$\int_{t_0}^{t} \int_V (T-U)dVdt$$

where T is the kinetic energy density. Thus, in addition to the term,

$$F_k = n_\ell \frac{\partial U}{\partial u_{k,\ell}}$$

one can expect a similar term from T, i.e.,

$$P_k = \pm \frac{\partial T}{\partial \dot{u}_k} = \pm \rho\dot{u}_k$$

where ρ is the mass density and the $\pm$ denotes unit normals to the 'time' surfaces $t = t_0$ and $t = t_1$.

The constraint equations are

$$u_i = \bar{u}_i \quad \text{on } S_u \tag{6}$$

and

$$\delta u_i = 0 \quad \text{at} \quad t = t_0 \ .$$

The constraint at t_0 can be satisfied by specifying u_i at t_0 but this cannot be done for the later time t_1 in the ordinary initial value problem. Thus, Hamilton's Principle contains at least one non-holonomic constraint.

Further, one notes that both momentum and displacement are prescribed on the <u>same</u> <u>surface</u> t_0. Thus, the quantity

$$\bar{P}_k \delta u_k]_{t_0} \ , \quad \text{unlike} \quad \bar{F}_k \delta u_k]_{S_f} \ ,$$

does not appear in Hamilton's Principle and analogous definitions for the λ_i are therefore not available. While the pressure of non-holonomic constraints can be handled by a more general Lagrange multiplier procedure [9], the overlapping of surfaces on which displacement and

momentum are specified proves insurmountable in applying the Lagrange multiplier technique. One concludes, therefore, that straightforward use of the Lagrange multiplier technique to completely unconstrain Hamilton's Principle in the time domain is not possible.

III. INITIAL VALUE PROBLEMS AND ADJOINT VARIATIONAL PRINCIPLES. The work of the previous section demonstrates that there is no difficulty in using the Lagrange multiplier method in the space domain where the governing equations are elliptic but only in the time domain where hyperbolic systems are encountered. In this section it is shown that hyperbolic systems can also be treated by the multiplier method provided consideration is given not only to the physical system but also its adjoint. This is most easily shown by example.

Consider the following initial value problem:

$$u'' + u' + u = 0 \quad ; \quad 0<x<1$$

$$u(0) = u'(0) = 0 \; . \tag{7}$$

The adjoint to the system Eq. (7) is

$$v'' - v' + v = 0$$

$$v(1) = v'(1) = 0 \; . \tag{8}$$

An adjoint variational principle may be found by multiplying Eq. (7) by the adjoint variable v and integrating over the domain. Thus:

$$\int_0^1 v(u''+u'+u)dx - [vu']_0^{\ell} = I \equiv \int_0^1 (uv+u'v-u'v')dx \; . \tag{9}$$

If the variation of I is made to vanish

$$\delta I = 0 = [v\delta u-u'\delta v-v'\delta u]_0^1 + \int_0^1 (u''+u'+u)\delta v dx + \int_0^1 (v''-v'+v)\delta u dx . \tag{10a}$$

Now if v(1) and u(0) are specified a priori,

$$\text{i.e.,} \quad \begin{aligned} v(1) &= 0 \\ u(0) &= 0 \end{aligned} \tag{10b}$$

then,

$$\delta I = 0 = u'(0)\delta v(0) - v'(1)\delta u(1) + \int_0^1 L(u)\delta v dx + \int_0^1 L^*(v)\delta u dx. \quad (10c)$$

Since only $\delta v(1)$ and $\delta u(0)$ are constrained to vanish in Eq. (10c), the rest of the variations are independent and arbitrary. Thus in addition to Eq. (10b) we have

$$\begin{aligned} L(u) &\equiv u'' + u' + u = 0 \\ L^*(v) &\equiv v'' - v' + v = 0 \\ u'(0) &= 0 \\ v'(1) &= 0 \ . \end{aligned} \quad (10d)$$

Equations (10b) and (10d) comprise the entire system of equations for the adjoint and physical systems. As it stands, Eq. (10c) is a constrained adjoint variational principle since the constraints (Eq. (10b)) are imposed a priori. The fact that Eq. (10c) does not, by itself, yield <u>all</u> of the initial conditions illustrates its constrained character. However, adding the constraints Eq. (10b) to this variational statement via the Lagrange multiplier method will free the principle of constraints and all of the initial conditions as well as the differential equations are then regurgitated. Thus

$$\delta\{\int_0^1 (uv+u'v-v'u')dx + \lambda_1 u(0) + \lambda_2 v(1)\} = 0$$

$$= - u'\delta v\Big|_0^1 + \int_0^1 (u''+u'+u)\delta v dx + (v-v')\delta u\Big|_0^1$$

$$+ \int_0^1 (v''-v'+v)\delta u dx + \lambda_1\delta u(0) + u(0)\delta\lambda_1 + \lambda_2\delta v(1) + v(1)\delta\lambda_2. \quad (11)$$

$\delta u(0)$ and $\delta v(1)$ may be <u>viewed</u> as arbitrary if

$$\begin{aligned} \lambda_1 &\equiv v(0) - v'(0) \\ \lambda_2 &\equiv u'(1) \ . \end{aligned} \quad (12)$$

Substituting the definitions (Equations (12)) into equation (11) and integrating by parts:

$$\int_0^1 L(u)\delta v dx + \int_0^1 L^*(v)\delta u dx + u'(0)\delta v(0)$$

$$+ (v(1)-v'(1))\delta u(1) - u(0)\delta v'(0) + v(1)\delta u'(1) = 0 \qquad (13)$$

Since all variations are now viewed as arbitrary,

$$L(u) \equiv u'' + u' + u = 0 \quad ; \quad 0<x<1$$

$$u(0) = 0$$

$$u'(0) = 0 \qquad (14a)$$

$$L(v) \equiv v'' - v' + v = 0 \quad ; \quad 0<x<1$$

$$v(1) = 0$$

$$v(1) - v'(1) = -v'(1) = 0. \qquad (14b)$$

Thus under the definitions (Equations (12)), equation (11) becomes an unconstrained adjoint variational principle which corresponds to the physical system together with its adjoint,

i.e., $$\delta\{\int_0^1 (uv+u'v-v'u')dx + (v(0)-v'(0))u(0) + u'(1)v(1)\} = 0 \qquad (15)$$

In view of the linearity of the systems considered and the arbitrariness of δu and δv, the portion of equation (13) which yields the u-system may be considered separately from that which gives the v-system,

i.e., $$\int_0^1 (u''+u'+u)\delta v dx + u'(0)\delta v(0) - u(0)\delta v'(0) = 0. \qquad (16)$$

Further, there is no reason why the variations on v cannot be those performed on u. Thus,

$$\int_0^1 (u''+u'+u)\delta u dx + u'(0)\delta u(0) - u(0)\delta v'(0) = 0 \,. \qquad (17)$$

One notes in passing that equation (17), unlike equation (15) is not of the form $\delta I = 0$, that is, no functional exists unless the adjoint system is included. Thus equation (17) might be more properly called a variational 'statement' as opposed to a 'principle'.

The Lagrange multiplier technique may be applied to the more general equations governing the motion of a linearly elastic solid. For example, consider the following system:

$$\frac{\partial U}{\partial u_k} + \rho \ddot{u}_k - \sigma_{\ell k,\ell} = 0$$

$$u_k - \bar{u}_k(t) = 0 \quad \text{on } S_u$$

$$- n_\ell \sigma_{\ell k} + \bar{F}_k = \text{on } S_f$$

$$u_k - \bar{u}_k(t_0) = 0 \quad ; \quad t = t_0$$

$$\dot{u}_k - \bar{v}_k = 0 \quad ; \quad t = t_0 \,. \tag{18}$$

The result of applying the Lagrange multiplier technique to this system and its adjoint is the following variational statement

$$\begin{aligned} 0 = \int_{t_0}^{t_1} &[\int_V \delta L dV + \int_{S_N} \bar{F}\delta u_k dS + \int_{S_u} \delta\{n_\ell \sigma_{\ell k}(u_k - \bar{u}_k)\} dS] dt \\ &+ \int_V dV\{-\dot{u}_k(t_1)\delta u_k(t_1) + \bar{v}_k \delta u_k(t_0) \\ &+ [u_k(t_0) - \bar{u}_k(t_0)]\}\rho\delta\dot{u}_k(t_0) \end{aligned} \tag{19}$$

where L is the Lagrange density

$$L = 1/2\rho\mathring{u}_k\dot{u}_k - U(u_k, u_{k,\ell}, x_j) .$$

Equation (19) and the result obtained by Tiersten [6] are identical except that in the interest of simplicity, no material surface of discontinuity has been considered in Eq. (19). It is to be noted that all variations are unconstrained so that the trial functions used in seeking an approximate solution need not satisfy any boundary or initial conditions a priori. If trial functions can be chosen that do satisfy some of the boundary constraints beforehand, convergence will usually be more rapid.

IV. APPLICATIONS.

Example 1: Wave Equation

$$S \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = 0$$

$$u(0,t) = g_0(t), \qquad u(\ell,t) = g_1(t)$$

$$u(x,0) = h_0(x), \qquad \dot{u}(x,0) = h_1(x) \tag{20}$$

Thus,

$$U = 1/2S(u')^2 \qquad \bar{F} \equiv 0$$

$$\sigma = \frac{\partial U}{\partial u'} = S_u{}'$$

Substituting the boundary conditions and the expressions for U and σ into Eq. (19) results in the following variational statement:

$$0 = \int_0^{t_1} \{ \int_0^{\ell} (a^2 u \delta u - u'\delta u')dx + [u(\ell,t)$$

$$-g_1(t)]\delta u'(\ell,t) - [u(0,t) - g_0(t)]\delta u'(0,t) + u'(\ell,t)\delta u(\ell,t)$$

$$-u'(0,t)\delta u(0,t)\}dt + \int_0^{\ell} a^2[-\dot{u}(x,t_1)\delta u(x,t_1)$$

$$+h_1(x)\delta u(x,0) + [u(x,0) - h_0(x)]\delta\dot{u}(x,0)]dx \tag{21}$$

where $a^2 = \rho/S$. A matrix formulation of Eq. (21) is achieved by substituting the approximation

$$u(x,t) = \sum_{j=1}^{NxN} a_j(x,t)c_j$$

as in the Ritz procedure but without any constraint requirements on the a_j set a priori. The result is a set of algebraic equations of the form

$$\sum_{j=1}^{NxN} k_{ij} c_j = r_i \qquad i=1,NxN \tag{22}$$

for the determination of the constants $c_j(\ell,t_1)$. Results are given in Tables 1 and 2 for the case

$$g_0 = g_1 = h_1 = 0; \quad h_0(x) = \sin\pi x/\ell; \quad \ell = 2 .$$

The shape functions $a_j(x,t)$ are taken to be products of polynomials in x and t. Good convergence is obtained for N = 8. As expected, Tables 1 and 2 show a decline in accuracy as the interval of integration is doubled.

Example 2: Free Oscillator-Particle Mechanics

$$\ddot{u} + \omega^2 u = 0, \quad u(0) = u_0, \quad \dot{u}(0) = v_0 \tag{23}$$

The variational formulation for this problem is

$$\int_0^{t_1} (\dot{u}\delta\dot{u} - \omega^2 u\delta u)dt - \dot{u}(t_1)\delta u(t_1) + v_0\delta u(0)$$
$$+ u(0)\delta\dot{u}(0) - u_0\delta\dot{u}(0) = 0 . \tag{24}$$

Table 3 gives the results for the case $u_0 = 0$, $v_0 = \omega = 2\pi$, and $t_1 = 1$. The assumed shape functions are polynomials in the time variable. A polynomial of order eight again gives good convergence.

Example 3: Response of a Beam to a Moving Mass

A concentrated mass is assumed to move at constant velocity v along the length of a uniform Euler beam, simply supported at each of its ends and having zero displacement and velocity at time t = 0. Under suitable definitions for k and m, the representative equations may be written [10]

$$y^{iv} + k\ddot{y} + f(x,t) = 0$$

$$y(0,t) = y''(0,t) = y(1,t) = y''(1,t) = 0$$

$$y(x,0) = \dot{y}(x,0) = 0 . \tag{25}$$

The function f(x,t) consists of a sum of inertial terms

$$f(x,t) = m(\ddot{y}+2v\dot{y}'+g''v^2y'')\delta(x-vt)$$

where g denotes the gravitational constant and δ is the Dirac function. The appropriate variational equation is

$$\int_0^{t_1}\int_0^1 (y''\delta y''-k\dot{y}\delta\dot{y}+f(x,t)\delta y)dxdt$$

$$+k\int_0^1 \{\mathring{y}(x,t_1)\delta y(x,t_1) - y(x,0)\delta\dot{y}(x,0)\}dx = 0 \ . \qquad (26)$$

A matrix approximation to Eq. (26) is obtained as in the first example, again using products of polynomials through order eight. The results are shown in Figure 1 as a comparison with values scaled from the experimental curves of Ayre, Jacobsen, and Hsu [11] for the case v = v*/4, v* being the lowest velocity to produce resonance when the load is a moving weight only. The magnitude assigned to the moving mass is 25% of the total mass of the beam of length L. The displacements have been normalized with respect to the maximum deflection produced if the weight was applied statically at midspan. This problem has also been treated previously by the author [10], using a conventional finite element method resulting in a set of differential equations in time. The numerical integration of these equations appeared to require a considerably longer computation time.

V. CONCLUSIONS. The unconstrained variational statement first developed and used by Tiersten for the solution of field displacements within a body containing a surface of discontinuity can indeed yield solutions to boundary-initial value problems. Further, the variational statement from which such solutions are possible can be formally constructed by the Lagrange multiplier method if the adjoint system is also considered.

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Table 1 Solution to wave equation $0 \le x \le 2.0$; $0 \le t \le 2.0$ (exact values in parentheses)

t/x	0.0	0.4	0.8	1.0	1.2	1.6	2.0
0.0	.000060 (.000000)	.587782 (.588785)	.951066 (.951057)	1.000000 (1.000000)	.951063 (.951057)	.587785 (.587785)	.000056 (.000000)
0.4	.000014 (.000000)	.475532 (.475528)	.769422 (.769421)	.809006 (.809017)	.769421 (.769421)	.475533 (.475528)	.000013 (.000000)
0.8	−.000001 (.000000)	.181636 (.181636)	.293889 (.293893)	.309010 (.309017)	.293890 (.293893)	.181635 (.181636)	.000001 (.000000)
1.0	−.000003 (.000000)	.000000 (.000000)	−.000002 (.000000)	−.000001 (.000000)	−.000001 (.000000)	−.000001 (.000000)	−.000002 (.000000)
1.2	−.000000 (.000000)	−.181637 (−.181636)	−.293889 (−.293893)	−.309010 (−.309017)	−.293890 (−.293893)	−.181636 (−.181636)	−.000001 (−.000000)
1.6	−.000013 (.000000)	−.475531 (−.475528)	−.769420 (−.769421)	−.809006 (−.809017)	−.769420 (−.769421)	−.475531 (−.475528)	−.000012 (−.000000)
2.0	−.000036 (.000000)	−.587786 (−.587785)	−.951055 (−.951057)	−.999986 (−1.000000)	−.951056 (−.951057)	−.587785 (−.587785)	−.000033 (−.000000)

Table 2 Solution to wave equation $0 \le x \le 2.0$; $0 \le t \le 4.0$ (exact values in parentheses)

t/x	0.0	0.4	0.8	1.0	1.2	1.6	2.0
0.0	.000037 (.000000)	.595863 (.587785)	.964161 (.951057)	1.013774 (1.000000)	.964161 (.951057)	.595867 (.587785)	.000031 (.000000)
0.8	.000008 (.000000)	.183947 (.181636)	.297637 (.293893)	.312952 (.309017)	.297638 (.293893)	.183948 (.181636)	.000007 (.000000)
1.6	−.000014 (.000000)	−.477206 (−.475528)	−.772139 (−.769421)	−.811866 (−.809017)	−.772139 (−.769421)	−.477207 (−.475528)	−.000012 (−.000000)
2.0	−.000013 (.000000)	−.587942 (−.587785)	−.951308 (−.951057)	−1.000253 (−1.000000)	−.951309 (−.951057)	−.587943 (−.587785)	−.000013 (−.000000)
2.4	−.000006 (.000000)	−.474475 (−.475528)	−.767707 (−.769421)	−.807204 (−.809017)	−.767707 (−.769421)	−.474475 (−.475528)	−.000007 (−.000000)
3.2	−.000001 (.000000)	.181191 (.181636)	.293165 (.293893)	.308247 (.309017)	.293166 (.293893)	.181191 (.181636)	.000001 (.000000)
4.0	.000002 (.000000)	.587756 (.587785)	.951014 (.951057)	.999945 (1.000000)	.951014 (.951057)	.587755 (.587785)	.000005 (.000000)

Table 3 Solution to free oscillator problem $0 \le t \le t_f = 1.0$

t	Computed solution	Exact solution
0.0	.00305	0.00000
0.1	.58656	.58779
0.2	.95218	.95106
0.3	.95159	.95106
0.4	.58670	.58779
0.5	−.00058	0.00000
0.6	−.58704	−.58779
0.7	−.95058	−.95106
0.8	−.95147	−.95106
0.9	−.58775	−.58779
1.0	−.00001	0.00000

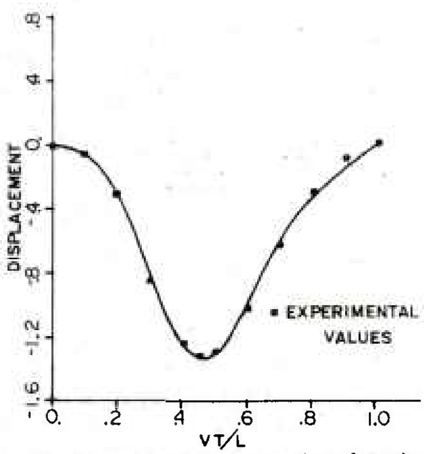

Fig. 1 Displacement of beam at location of moving mass.

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